

# How Often Do Diquarks Form? A Very Simple Model

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Starting from a textbook result, the nearest-neighbor distribution of particles in an ideal gas, we develop estimates for the probability with which quarks  $q$  in a mixed  $q, \bar{q}$  gas are more strongly attracted to the nearest  $q$ , potentially forming a diquark, than to the nearest  $\bar{q}$ . Generic probabilities lie in the range of tens of percent, with values in the several percent range even under extreme assumptions favoring  $q\bar{q}$  over  $qq$  attraction.

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## I. INTRODUCTION

The observation of multiple heavy-quark exotics in recent years, starting with the Belle discovery of the presumptive  $qq\bar{q}\bar{q}$  state  $X(3872)$  in 2003 [1], has provided entirely new opportunities for developing a deeper understanding of the QCD dynamics responsible for binding quarks into color-singlet hadrons. While it is mathematically true that all  $SU(3)_c$  color singlets assembled from quarks can be decomposed as products of the  $qqq$  and  $q\bar{q}$  combinations familiar from conventional baryons and mesons, respectively,<sup>1</sup> group theory alone does not dictate the nature of structures dynamically generated within hadrons.

Several theoretical pictures have been advocated to describe the multi-quark exotics. The mathematical feature of  $SU(3)_c$  just described encourages one to consider *molecules* of color-singlet hadrons, and heavy-quark models of this sort have been contemplated for almost the entire history of QCD [2, 3]. Alternately, in *hadroquarkonium* [4], the heavy  $q\bar{q}$  pair lies at the center of a cloud generated by the lighter quarks. In *diquark* models, popularized for light-quark systems in Refs. [5, 6] and for the new heavy quarkoniumlike exotics like  $X(3872)$  in Ref. [7], diquark  $qq$  and  $\bar{q}\bar{q}$  pairs form via the color attractive channel  $\mathbf{3} \times \mathbf{3} \rightarrow \mathbf{\bar{3}}$  and its conjugate, as discussed below. The *dynamical diquark* picture [8, 9] further purports that the diquarks do not act as components of stable molecules, but rapidly separate until confinement forces the system to hadronize. And in *kinematic-effect* models, first suggested for the new exotic states in Ref. [10], the opening of hadronic thresholds can generate structures resembling resonances nearby in mass, either by themselves or by coupling to other channels.

In this work we are interested in diquarks, and particularly the relative rate at which  $qq$  (or  $\bar{q}\bar{q}$ ) pairs form compared to the rate for  $q\bar{q}$  pairs. There exists, after all, no universally accepted experimental evidence for the

existence of diquarks, so that one might suspect their formation to be a rather rare occurrence. Nevertheless, fundamental QCD considerations suggest otherwise. The color dependence of the short-distance coupling of elementary particles in  $SU(3)_c$  representations  $R_1$  and  $R_2$  to the product representation  $R$  is proportional to the combination

$$\mathcal{C}(R, R_1, R_2) \equiv C_2(R) - C_2(R_1) - C_2(R_2), \quad (1)$$

where  $C_2$  is the representation's quadratic Casimir. For  $qq$  or  $q\bar{q}$  systems, one finds the relative size of the couplings from Eq. (1) to be

$$\mathcal{C}(R, R_1, R_2) = \frac{1}{3}(-8, -4, +2, +1) \text{ for } R = (\mathbf{1}, \mathbf{\bar{3}}, \mathbf{6}, \mathbf{8}), \quad (2)$$

respectively. As one might expect, the strongest coupling is that of the color-singlet  $q\bar{q}$  combination, which provides a direct route for the formation of mesons. However, the aforementioned diquark coupling is a full one-half as strong at short distance, while the two repulsive channels are both smaller than either of the attractive ones.

Still, one might expect that in a  $qq\bar{q}\bar{q}$  system, even if diquarks initially form, the greater attraction in the  $q\bar{q}$  singlet channel suggests that the system subsequently rearranges itself into a two-singlet combination. This issue is less acute in the dynamical diquark picture, in which the diquarks achieve substantial separation before this rearrangement can occur. Nevertheless, the attraction described by Eq. (2) strictly holds only at short distances, where single-gluon exchanges dominate. One expects the interaction between a particular  $qq$  or  $q\bar{q}$  pair at separations beyond a few tenths of a fm to be heavily screened by gluon and sea-quark pair creation.

The question of whether diquarks actually appear as important hadronic substructures then can be discussed in terms of the exact nature of the spatial distribution of the nearest neighbors of the quarks in the production process. Statistically speaking, with some finite probability a  $q$  will find itself much closer to another  $q$  than to a  $\bar{q}$ , which allows the diquark attraction to dominate. But pure spatial proximity cannot be the whole story, however, since in weak and electromagnetic decays the created  $q\bar{q}$  pair automatically forms a color singlet, and

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<sup>1</sup> If valence gluons  $g$  are included, then hybrids  $q\bar{q}g$  and glueballs  $gg, ggg$ , etc. complete the list of color-singlet substructures.

yet does not always by itself form a single meson (the so-called color-suppressed decay diagrams give one counterexample); a large initial relative momentum between quarks can apparently overwhelm the proximity effect in certain circumstances, and hadronization can be delayed until after the quarks lose a significant amount of energy.

We therefore attempt to remove such complications and create a toy model as simple as possible. We generalize the system of the handful of quarks and anti-quarks created in a typical decay or collider process—whose numbers are exactly equal in a meson decay,  $e^+e^-$ , or  $p\bar{p}$  collision, or differ only by 6 in a  $pp$  collision—as forming a  $(q, \bar{q})$  gas of arbitrarily large extent, and assume that the particles have become essentially static (or at least achieve something resembling a low-temperature thermal distribution) prior to hadronization. Indeed, we model the system as a two-component ( $q$  and  $\bar{q}$ ) ideal gas and ask a very basic question: With what probability is a given  $q$  preferentially attracted—at least initially—to another  $q$  rather than a  $\bar{q}$ ? We show that this probability could be as large as tens of percent, and indeed is difficult to reduce to lower than a few percent. Diquarks should be common components in hadronic processes.

If one subscribes to the long-studied idea [11] that all baryons have a significant diquark component, then the simple model studied here also provides a first step to addressing the relative rates of meson, baryon, and tetraquark production.

It is important to point out that other works discussing diquark production focus on different energy, density, or temperature regimes. Direct diquark production is built into jet fragmentation event generators dating back at least as far as the famous Lund model [12]; however, the diquark attraction described here is of the lower-energy “non-prompt” variety. Diquark condensation in dense QCD is a well-studied phenomenon (*e.g.*, [13]), and has been extended also to finite temperature [14]; obviously, an ideal low-temperature gas is neither of these. In addition, in more formal work using an effective supersymmetric embedding of quantum mechanics into AdS space [15, 16], diquarks are found to be absolutely natural and indeed essential hadronic components: In particular, the  $qq$  and  $q\bar{q}$  attraction strengths turn out to be the same, and baryons are naturally quark-diquark bound states.

This paper is organized as follows. In Sec. II, the problem of nearest neighbors in an ideal gas is treated in both the original textbook case and the two-component case. Section III applies these ideas to the case of diquark attraction, and we outline the many reasons why this treatment falls far short of real QCD, as well as the ways in which the model attempts to address at least some of them. Explicit model calculations appear in Sec. IV, where we find that diquark attraction should be a rather common occurrence in hadronic physics. Section V summarizes and concludes.

## II. NEAREST-NEIGHBOR DISTANCES IN IDEAL GASES

### A. The Classic Problem

We begin with a standard textbook problem, that of the distribution of nearest-neighbor particles randomly (Poisson) distributed to form an ideal gas. This problem was first addressed by P. Hertz in 1909 [17]. We present here the elegant derivation by Chandrasekhar [18], as it is useful both for establishing notation and for being amenable to straightforward generalizations.

In the original problem, the gas particles are classical, pointlike, and noninteracting except through possible elastic collisions, and (implicitly) obey Maxwell-Boltzmann statistics. The system effectively is infinite in extent and isotropic, so that any point may be treated as typical.

We are interested in the radial probability density  $w(r)$  of the nearest particle to the (arbitrary) origin to lie at a distance  $r$ , which satisfies the normalization condition

$$\int_0^\infty dr w(r) = 1, \quad (3)$$

noting that, in light of the isotropy assumption, the angular integrals and volume-element  $r^2$  have already been absorbed into the definition of  $w(r)$ . We are also interested in the mean nearest-neighbor distance,

$$\langle r \rangle \equiv \int_0^\infty dr r w(r). \quad (4)$$

Let  $n$  be the volume number density of particles. Then  $w(r)$  satisfies

$$w(r)dr = \left[ \int_r^\infty dr' w(r') \right] n \cdot 4\pi r^2 dr. \quad (5)$$

In words, Eq. (5) says: The probability for the nearest neighbor to the origin to lie in a spherical shell of radius  $dr$  at  $r$  [which is  $w(r)dr$ ] equals the product of the probability that no particle lies between the origin and radius  $r$ —*i.e.*, that the nearest neighbor lies between  $r$  and  $\infty$  (which is the integral)—times the number of particles in the spherical shell (which is  $n \cdot 4\pi r^2 dr$ ). Equation (5) may be recast as a differential equation,

$$\frac{d}{dr} \left[ \frac{w(r)}{4\pi r^2 n} \right] = -4\pi r^2 n \left[ \frac{w(r)}{4\pi r^2 n} \right]. \quad (6)$$

The original problem assumes  $n$  constant (appropriate to a uniform ideal gas of infinite extent), but for later use, let us generalize to the case of a radially dependent density  $n(r)$  (which implies a unique origin  $r = 0$  for the system). The solution to Eq. (6) is

$$w(r) = C \cdot 4\pi r^2 n(r) \exp \left[ -4\pi \int_0^r dr' n(r') r'^2 \right]. \quad (7)$$

The integration constant  $C$  is fixed by imposing the normalization condition Eq. (3). One finds

$$C = \left\{ 1 - \exp \left[ -4\pi \int_0^\infty dr n(r) r^2 \right] \right\}^{-1}. \quad (8)$$

In the case of constant  $n$ , or at least  $n(r)$  that remains sufficiently large as  $r \rightarrow \infty$ , the exponential approaches zero, and  $C \rightarrow 1$ .

In the original case of constant  $n$ , the integrals may be performed analytically, giving

$$w(r) = 4\pi r^2 n \exp\left(-\frac{4\pi r^3}{3} n\right) = \frac{3}{a} \left(\frac{r}{a}\right)^2 \exp\left[-\frac{r^3}{a^3}\right], \quad (9)$$

$$\langle r \rangle = \Gamma\left(\frac{4}{3}\right) \left(\frac{4\pi n}{3}\right)^{-\frac{1}{3}} = \Gamma\left(\frac{4}{3}\right) a \simeq 0.8928a, \quad (10)$$

where the density  $n$  defines the natural length scale,  $a \equiv (4\pi n/3)^{-1/3}$ . Indeed, one may also show that  $\langle r^3 \rangle = a^3$ .

## B. Two-Component Ideal Gas

We now extend the previous derivation to solve another simple classical problem, which to our knowledge has not previously been addressed: Consider an ideal gas consisting of two species, 1 and 2, with corresponding number densities per unit volume (natural length scales)  $n_i$  ( $a_i$ ),  $i = 1, 2$ . Starting from some fiducial point (the origin), what is the probability that the nearest particle of type 2 does not appear until a distance  $k$  times further than that at which the nearest particle of type 1 appears? We have in mind of course that the origin contains a quark, and want to know the statistical likelihood that the nearest neighbor happens to be a quark rather than an antiquark, by a chosen distance ratio  $k$ . Note that the ratio  $k$  can lie anywhere in  $(0, \infty)$ , and that the specification of the species at the origin ( $q$  or  $\bar{q}$ ) need only be made at the end of the calculation.

The probability for the first particle of type  $i$  lying at a distance  $r_i$  from the origin is  $w_i(r_i)dr_i$ , where  $w_i$  is simply the function in Eq. (7) or (9) defined with  $n \rightarrow n_i$ . Since the gas is ideal, the probabilities are independent, and the combined probability that the first particle of type 1 lies at distance  $r_1$  and the first particle of type 2 lies at distance  $r_2$  from the origin is

$$p(r_1, r_2) = w_1(r_1)dr_1 w_2(r_2)dr_2. \quad (11)$$

Next, the probability  $P(r_1, k)$  that the first particle of type 1 lies at a distance  $r_1$  from the origin and that the first particle of type 2 lies at least a distance  $kr_1$  from the origin is then

$$P(r_1, k) = \int_{r_2=kr_1}^\infty p(r_1, r_2) = w_1(r_1)dr_1 \int_{kr_1}^\infty w_2(r_2)dr_2, \quad (12)$$

and the probability  $P_{1,2}(k)$  that the first particle of type 2 lies at a distance  $k$  times that of the first particle of type 1, regardless of the specific value of  $r_1$ , is

$$P_{1,2}(k) = \int_{r_1=0}^\infty P(r_1, k) = \int_0^\infty w_1(r_1)dr_1 \int_{kr_1}^\infty w_2(r_2)dr_2. \quad (13)$$

This expression, with inputs suitably chosen, is used in this paper to calculate likelihoods relevant to the formation of diquarks versus color-singlet  $q\bar{q}$  pairs.

In the case of constant densities  $n_i$ , the integrals in Eq. (13) can be performed in closed form. Since

$$\int_R^\infty dr_i w_i(r_i) = e^{-(R/a_i)^3}, \quad (14)$$

one finds

$$P_{1,2}(k) = \frac{a_2^3}{k^3 a_1^3 + a_2^3} = \frac{n_1}{k^3 n_2 + n_1}. \quad (15)$$

Several limits of this simple expression are easy to understand. It must vanish as  $k \rightarrow \infty$ , which is the unlikely case that all type-2 particles are arbitrarily far from the origin; indeed, the scaling with  $1/k^3$  is expected from the volume effect of scaling the distance ratio as  $k$ . The limit  $k \rightarrow 0$  simply means the case where the nearest type-2 particle can be anywhere, for which  $P_{1,2}(k)$  must approach unity. The case  $k = 1$  means the probability of finding the first type-2 particle to lie at least as far from the origin as the first type-1 particle; if  $n_1 = n_2$ , one expects the probabilities for either of the species to provide the nearest particle to the origin to be equal:  $P_{1,2}(1) = \frac{1}{2}$ , exactly as one finds from Eq. (15).

## III. IDEALIZED DIQUARK ATTRACTION

### A. Warmup: Electric Charges

Using the analysis of the two-component ideal gas in the previous section, let us begin by considering an analogous problem with static electric charges. Of course, even the particles of an ideal gas at finite temperature have a nontrivial velocity distribution, meaning that the static assumption is already suspect, and Earnshaw's theorem moreover forbids such a static system from being in a stable equilibrium. Nevertheless, one can take the system at the initial time  $t = 0$  to start from rest and assume that the dominant interaction between charges is the central Coulomb force. The model two-component gas consists of type-1 particles of charge  $+\frac{1}{2}$  and type-2 particles of charge  $+1$ , designed to emulate the factor-2 difference between the short-distance strength of the  $qq$  and  $q\bar{q}$  channels. A negative test charge, attracted to both charge species, is placed at the origin. How frequently does its initial attraction to a type-1 particle equal or exceed that to a type-2 particle, due only to the initial distribution of the particles?

Several limitations of addressing the problem in this way have already been noted, but let us remark in addition that initial attraction to a neighbor is not the same as the formation of a compact state with this neighbor. The collective effect of several other neighbors can overwhelm it, the  $t > 0$  migration of the test charge towards its most attractive neighbor disturbs the initial configuration (as does the movement of the other charges), allowing other charges to disrupt the initial attraction, and of course the proper treatment of charges in motion requires one to include the effects of magnetic fields.

Nevertheless, the idealized problem is well defined. Since the Coulomb attraction obeys an inverse-square law, a type-1 particle provides the most attractive initial interaction for  $k = \sqrt{2}$  (i.e., the nearest type-2 particle lies at least  $\sqrt{2}$  times farther from the origin than the nearest type-1 particle). Assuming that the densities of the two types of particles are equal and constant, Eq. (15) gives that the test charge is initially more attracted to the smaller type-1 charge with probability  $P_{1,2}(\sqrt{2}) = 1/(2\sqrt{2} + 1) \simeq 26\%$ , a sizeable fraction.

## B. Unscreened Quarks

An ideal, initially stable distribution of  $q$  and  $\bar{q}$  is much more complicated than the example just discussed for several reasons. First, only the bare (short-distance) quark interaction obeys the simple scaling from Eq. (2) in which the attractive  $qq$  channel has half the strength of the attractive  $q\bar{q}$  channel. At larger distances, the interactions represented by the exchange of colored gluons (and additional  $q\bar{q}$  pair creation) serve to screen the bare interaction; we attempt a simple-minded modeling of this effect below. It is worth mentioning that this “Casimir scaling” given by Eq. (1) is violated only at three-loop perturbative order [19] and its existence to as much as  $r \sim 1$  fm is well supported in lattice simulations [20].

Second, as noted in the Introduction, the short-distance interactions also feature repulsive  $qq(\mathbf{6})$  and  $q\bar{q}(\mathbf{8})$  combinations. One could certainly extend the derivation of the previous section to a 3- or 4-component gas, but one sees from Eq. (2) that the short-distance  $qq\text{-}\mathbf{6}$  repulsion is only  $\frac{1}{2}$  as large as the  $\mathbf{3}$  attraction, and the  $q\bar{q}\text{-}\mathbf{8}$  repulsion is only  $\frac{1}{8}$  as large as the  $\mathbf{1}$  attraction. Moreover, we are interested in the attractive forces that ultimately lead to quarks combining into hadrons, and therefore ignore the effect of the repulsive channels (which, presumably, lead to the separation of quark clusters into hadrons). However, even under the assumption of neglecting repulsive forces, these channels have an important effect: A generic  $qq$  has 9 possible color combinations, of which only the 3 forming the  $\mathbf{3}$  are attractive, while for the 9 possible color combinations of a generic  $q\bar{q}$  pair, only the singlet  $\mathbf{1}$  combination is attractive. If one assumes equal densities of  $q$  and  $\bar{q}$ , then the *effective* density of attractive  $q$ ’s is 3 times the effective density of attractive  $\bar{q}$ ’s, i.e.,  $n_1 = 3n_2$ . Again using  $k = \sqrt{2}$ , one

finds from Eq. (15) the remarkable result

$$P_{1,2}(n_1 = 3n_2; k = \sqrt{2}) = 3(3 - \sqrt{2}) \simeq 51.5\%, \quad (16)$$

suggesting a very substantial probability for diquark attraction in the case of unscreened quark color charges.

The assumption that the initial interaction is dominated by the static color-Coulomb force relies on the nonrelativistic expansion of the quark bilinears  $\bar{q}\gamma^\mu q \rightarrow \delta_0^\mu q^\dagger q$  in the QCD Lagrangian; nonzero spatial momenta couple to  $\gamma^i$  and thus induce significant spin dependence in the interaction, especially for relativistic quarks. Relativity is also important for corrections to the assumption of a quark gas of infinite extent, since a fully correct treatment must include the retardation of the propagating interactions. On the other hand, since the exchange symmetry of  $\mathbf{3}$  (6) is antisymmetric (symmetric), the effect of the Pauli exclusion principle and neglecting the  $\mathbf{6}$  serves to exclude  $qq$  pairs in overall flavor-spin-space symmetric combinations. For distinct light ( $u$ ,  $d$ ) quarks in relative  $s$ -waves, isosinglet/spin-singlet and isotriplet/spin-triplet pairs survive this sorting, while for identical light quarks, the first combination is also excluded.

## C. Color Screening

Despite the significant number of exceptions and corrections to the ideal gas model thus far identified, the effects discussed above are still essentially classical. Real QCD is of course a quantum-mechanical theory, meaning that the concept of pointlike particles with potential interactions depending predominantly upon their separation, and hence the very concept of nearest neighbors, should be considered suspect. Quark wave functions have a finite extent; therefore, one’s first thought may be to reanalyze the nearest-neighbor derivation to include finite particle radii. A relevant calculation has been undertaken in Ref. [21] that models the particles as hard spheres, and presumably could be generalized to the case in which the spheres are partially penetrable “clouds”. In the context of the model here, such an effect could be incorporated by altering the functional dependence of the density functions  $n_{1,2}(r)$ . But even these modifications do not fully respect a fundamental feature of quantum mechanics: Interactions can collapse wave functions, which is natural in light of the fact that the fundamental QCD interaction  $\bar{q}(x)\gamma^\mu g_s A_\mu(x)q(x)$  is local. We therefore argue that the concept of nearest neighbors retains its significance when interpreted in the usual statistical sense of quantum mechanics.

The most important quantum-mechanical effect in modifying the ideal-gas picture, however, is a quantum field-theoretical effect: the crucial importance of color screening in strong interactions. The large size of the strong coupling  $\alpha_s = g_s^2/4\pi$  at low energies means large numbers of sea-quark  $q\bar{q}$  pairs created between the original quarks of the ideal gas, which serve to screen the

initial color interactions. Of equal significance, the non-Abelian nature of QCD means that the gluons themselves carry color charge, and hence self-couple and contribute to the color screening. Were the exact solutions to the renormalization group equations known at low scales, a rigorous treatment of the screening could be undertaken. In practical terms, the dominant feature of low-energy QCD is color confinement, whose effect we incorporate as an effective screening of color charges beyond a radius given by the largest typical hadron sizes,  $R = O(1 \text{ fm})$ .

Before examining the results of explicit model calculations, we summarize the multiple roles played by the density functions  $n_{1,2}(r)$ . We have used their overall scales to represent not only the relative densities of  $q$  and  $\bar{q}$ , respectively, but also the relative number of channels attractive to a test quark at the origin. Their functional dependences could in principle be used to model the finite extent of  $q$ ,  $\bar{q}$  wave functions, but here we use it to model the strong color screening in a  $q$ ,  $\bar{q}$  gas by introducing a characteristic screening radius  $R$ . The only other independent parameter in the model is the ratio  $k$ , indicating the relative distance at which a  $\bar{q}$  and  $q$  are attracted to the test quark at the origin with equal strength.

#### IV. EXPLICIT MODELS

Starting with the expressions Eqs. (7), (8), and (13), we model the effective screened densities using several plausible functional forms, and investigate the results for the probabilities  $P_{1,2}(k)$  as an indication of the likelihood of diquark attraction.

The three functional profiles are all chosen to have  $n_1 = 3n_2$  as discussed above, although altering the specific ratio of 3 does not alter the ultimate significance of the substantial values obtained for  $P_{1,2}(k)$ , as discussed below. The profiles are a hard-wall screen,

$$n_1^{(1)} = n_0 \Theta(R - r), \quad (17)$$

where  $\Theta$  is the Heaviside step function; a Saxon-Woods form with a skin depth  $d$ ,

$$n_1^{(2)} = n_0 \cdot \frac{1 + \exp(-\frac{R}{d})}{1 + \exp(-\frac{r-R}{d})}; \quad (18)$$

and a linear decrease out to the screening wall at  $R$ ,

$$n_1^{(3)} = n_0 \left(1 - \frac{r}{R}\right) \Theta(R - r); \quad (19)$$

all of which have the same central unscreened density  $n_0$ . For definiteness, we choose  $n_0 = (2/R)^3$  (indicating an expectation of encountering two  $q$ 's or  $\bar{q}$ 's before reaching  $R$ ) and  $d = R/2$ . We compare results for the three profiles and explore their  $k$  dependence in Table I. Note first that the result for the hard-wall screen  $n^{(1)}$  with  $k = \sqrt{2}$  and  $n_1 = 3n_2$  almost equals the result of Eq. (16), meaning that the color screening has little effect when  $k = \sqrt{2}$  and  $n_0 = (2/R)^3$ . In fact, Eq. (16) with

TABLE I: Values of  $P_{1,2}(k)$  in percent for the three screened density profiles of Eqs. (17)–(19), for three values of  $k$ .

	$k = \sqrt{2}$	$k = 2\sqrt{2}$	$k = 3\sqrt{2}$
$n_1^{(1)}$	50.29	10.85	3.46
$n_1^{(2)}$	53.24	13.99	4.79
$n_1^{(3)}$	40.40	9.04	3.04

TABLE II: Values of  $P_{1,2}(k = \sqrt{2})$  in percent for the screened density profile  $n_1^{(0)}$  of Eq. (17), for several values of  $n_0$ .

$n_0$	$1/(10R)^3$	$1/(5R)^3$	$1/(2R)^3$	$1/R^3$	$(2/R)^3$	$(3/R)^3$
$P_{1,2}(\sqrt{2})$	17.69	17.74	18.69	26.01	50.29	51.47

larger values of  $k$  continues to match the results in the first line of Table I quite well, within 10%. The Saxon-Woods form  $n^{(2)}$  with a substantial skin depth  $d = R/2$  actually gives somewhat larger values of  $P_{1,2}(k)$  than the hard-wall form  $n^{(1)}$ , due to the sampling of points for  $n^{(2)}$  with  $r > R$ . Of course, in the limit  $d \rightarrow 0$ , its profile Eq. (18) reduces to that of Eq. (17), which can also be checked numerically. Even a profile like  $n^{(3)}$  in Eq. (19), for which the effective density decreases to zero at  $r = R$ , decreases the values of  $P_{1,2}$  somewhat but leaves their order of magnitude intact. One can check that enhancing the rate of decrease even to a profile  $n(r)$  that falls exponentially fast (not exhibited here) does not fundamentally change this conclusion.

Besides the value of  $k$  and the shape of the profile function  $n(r)$ , the only remaining degrees of freedom in this model are the precise value of the central density  $n_0$  and the relative ratio  $n_1/n_2$  of  $q$  to  $\bar{q}$  channels attractive to the central quark, which was argued in the previous section to be 3. The value of  $n_0$  affects the results of the calculation due to its appearance in the exponentials of Eqs. (7)–(8); the main effect of changing its value is to change its precise relationship to the length scales in the problem, particularly to the screening radius  $R$ . For definiteness, consider the hard-wall profile  $n^{(1)}$  and  $k = \sqrt{2}$ . The results for several values of  $n_0$  are presented in Table II. We see that increasing  $n_0$  results in a rapid approach to the unscreened case of Eq. (16); and even a decrease of  $n_0$  to  $1/(10R)^3$  only results in a decrease of  $P_{1,2}(k)$  by a factor of 3, indeed with values approaching the asymptotic value  $1/2k^3$  (when  $k > 1$ ), as can be shown analytically from Eqs. (7), (8), and (13).

Lastly, with respect to the ratio  $n_1/n_2$ , one can check that the hard-wall density profile  $n^{(1)}$  of Eq. (17) gives results almost exactly matching the idealized unscreened formula Eq. (15). For example, if  $n_1/n_2$  is not 3 as suggested by simple color considerations, but somehow the effective density of attractive  $\bar{q}$ 's is 10 times this value, then  $P_{1,2}$  reduces from 50.29% to 9.59%, still a rather significant value. In every case, the probability  $P_{1,2}$  of a test quark being initially attracted most strongly to another  $q$  rather than a  $\bar{q}$  is at least several percent.

## V. CONCLUSIONS

We have seen that the large relative size of the short-distance attraction between quarks in the color-antitriplet channel compared to the attraction between a quark and an antiquark in the color-singlet channel leads inexorably to a given quark being initially attracted to a quark rather than an antiquark a sizeable fraction of the time. We interpret this initial attraction as the seed event in the formation of a compact diquark  $qq$  rather than a color-singlet  $q\bar{q}$  pair.

While the short-distance color attraction must be modified by QCD renormalization effects and the color-screening effects due to confinement, one still expects this attraction to extend out to some finite distance, and at greater distances still if both quarks and antiquarks are comparably screened. Under these assumptions, the probability of preferential  $qq$  attraction is in the tens of percent. Even if one allows the  $q\bar{q}$  attraction to somehow dominate by a factor of several at these distances (either through the relative force of attraction, modeled here by the parameter  $k$ , or through the relative effective density of  $\bar{q}$  compared to  $q$  attracted to the test quark, modeled here by  $n_1/n_2$ ), the probability of preferential  $qq$  attraction is still at least several percent. The key formula describing the most idealized situation, from which rule-of-thumb estimates may be obtained, is the unscreened result Eq. (15). In the context of the class of models described here, it is very difficult to completely suppress the  $qq$  attraction far below the level of  $q\bar{q}$  attraction.

What evidence, then, does one have of diquark production? While tantalizing hints of diquark substructure

have appeared in multiple light-quark systems in the past (such as the  $f_0$  and  $a_0$  mesons), the heavy-quark systems offer greater opportunities for disentangling clear signals because of the well-defined spectroscopy of heavy quarkonium systems. Consider one example, the charged  $J^P = 1^+$  exotic  $Z^-(4430)$ , confirmed at LHCb at high statistical significance [22], which is a prime candidate for a diquark-antidiquark system [8, 23], particularly if the diquarks are considered well separated, as in Ref. [8]. Its only measured decay mode thus far is  $Z^-(4430) \rightarrow \psi(2S)\pi^-$ , and its combined branching fraction in the decay chain  $B^0 \rightarrow Z^-(4430)K^+ \rightarrow \psi(2S)\pi^-K^+$  is [24]  $(6.0^{+3.0}_{-2.4}) \times 10^{-5}$ . In comparison, the branching fraction of  $B^0$  to the much lighter conventional  $1^{++}$  charmonium state  $\chi_{c1}$  (3511 MeV) and a  $K^0$  is  $(3.93 \pm 0.27) \times 10^{-4}$ , with similar values for other two-body charmonium decays. The branching fraction to  $\chi_{c1}$  is only a factor of 4–12 times larger, despite a two-body phase space that is 2.4 times larger. A large probability for diquark formation under these assumptions seems to be indicated.

Such positive signals from both data and from this simple model will hopefully encourage the development of much more realistic QCD treatments of  $qq$  attraction and diquark formation. It seems unavoidable that allowing for a substantial degree of diquark formation will need to be taken into account in future studies of high-energy and heavy-quark hadronic physics.

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